5. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = 2 - \frac{3}{z + \mathrm{i}} \qquad z \neq -\mathrm{i}$$

The half-line in the z-plane with equation $\arg z = \frac{\pi}{4}$ is mapped by T onto a circle C in the w-plane.

(a) Determine the equation of C in the form

$$(u - a)^2 + (v - b)^2 = k$$

where a, b and k are constants.

(7)

(b) Hence, or otherwise, determine

(i) the radius of C

(ii) the coordinates of the centre of C

(2)

(Total for Question 5 is 9 marks)

---

6. (a) Given that $z = \cos\theta + \mathrm{i}\sin\theta$, show that

(i) $\frac{1}{2}\left(z + \frac{1}{z}\right) = \cos\theta$

(ii) $\frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right) = \sin\theta$

(2)

(b) Hence show that

$$\sin^3\theta\cos^4\theta \equiv a\sin 7\theta + b\sin 5\theta + c\sin 3\theta + d\sin\theta$$

where a, b, c and d are constants to be determined.

(5)

(c) Use the result from part (b) and algebraic integration to determine the exact value of

$$\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^4\theta \,\mathrm{d}\theta$$

(4)

(Total for Question 6 is 11 marks)