Question Number	Scheme	Notes	Marks
5(a) **Way 1** **From LHS**	$\left(1-\text{sech}^2 x=\right)1-\left(\dfrac{2}{e^x+e^{-x}}\right)^2$	Replaces sech *x* with correct expression in terms of exponentials	B1
	$=\dfrac{\left(e^x+e^{-x}\right)^2-4}{\left(e^x+e^{-x}\right)^2}=\dfrac{e^{2x}+2+e^{-2x}-4}{\left(e^x+e^{-x}\right)^2}$	Expresses as a single fraction (or 2 fractions with the same denominator) and expands numerator	M1
	$=\dfrac{\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}=\tanh^2 x$	Fully correct proof	A1*
Way 2 **Diff. of 2 squares**	$1-\text{sech}^2 x=(1+\text{sech}\, x)(1-\text{sech}\, x)=\left(1+\dfrac{2}{e^x+e^{-x}}\right)\left(1-\dfrac{2}{e^x+e^{-x}}\right)$ Uses difference of two squares and replaces sech *x* with correct expression in terms of exponentials		B1
	$=\left(\dfrac{e^x+e^{-x}+2}{e^x+e^{-x}}\right)\left(\dfrac{e^x+e^{-x}-2}{e^x+e^{-x}}\right)=\dfrac{e^{2x}+1-2e^x+1+e^{-2x}-2e^{-x}+2e^x+2e^{-x}-4}{\left(e^x+e^{-x}\right)^2}$ Expresses as a single fraction and expands numerator		M1
	$=\dfrac{e^{2x}-2+e^{-2x}}{\left(e^x+e^{-x}\right)^2}=\dfrac{\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}=\tanh^2 x$	Fully correct proof	A1*
Way 3 **From RHS**	$\left(\tanh^2 x=\right)\dfrac{\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}$	Replaces tanh *x* with correct expression in terms of exponentials	B1
	$=\dfrac{e^{2x}-2+e^{-2x}}{\left(e^x+e^{-x}\right)^2}=\dfrac{e^{2x}+2+e^{-2x}}{\left(e^x+e^{-x}\right)^2}-\dfrac{4}{\left(e^x+e^{-x}\right)^2}$ Expands numerator and splits into two fractions		M1
	$=\dfrac{\left(e^x+e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}-\left(\dfrac{2}{e^x+e^{-x}}\right)^2=1-\text{sech}^2 x$	Fully correct proof	A1*
			(3)

Allow "meet in the middle" approaches as long as a conclusion is given e.g. lhs = rhs

Example:

$$rhs=\tanh^2 x=\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2} \quad \text{or} \quad lhs=1-\text{sech}^2 x=1-\left(\frac{2}{e^x+e^{-x}}\right)^2$$

B1: Replaces tanh *x* or sech *x* with a correct expression in terms of exponentials

$$\frac{\left(e^{2x}-1\right)^2}{\left(e^{2x}+1\right)^2}=\frac{e^{4x}-2e^{2x}+1}{e^{4x}+2e^{2x}+1} \quad \text{and} \quad 1-\left(\frac{2}{e^x+e^{-x}}\right)^2=\frac{\left(e^x+e^{-x}\right)^2-4}{\left(e^x+e^{-x}\right)^2}=\frac{e^{2x}-2+e^{-2x}}{e^{2x}+2+e^{-2x}}$$

M1: Makes progress by e.g. removing brackets on *rhs* and expressing *lhs* as a single fraction and expands numerator

$$\frac{e^{2x}-2+e^{-2x}}{e^{2x}+2+e^{-2x}}=\frac{e^{4x}-2e^{2x}+1}{e^{4x}+2e^{2x}+1}\Rightarrow 1-\text{sech}^2 x=\tanh^2 x$$

A1: Correct proof and (minimal) conclusion e.g. "= rhs" etc.

$$1-\operatorname{sech}^2 x = 1-\left(\frac{2}{e^x+e^{-x}}\right)^2 = \frac{e^{2x}+2+e^{-2x}-4}{\left(e^x+e^{-x}\right)^2} = \frac{e^{2x}+e^{-2x}-2}{e^{2x}+e^{-2x}+2} = \frac{\sinh^2 x}{\cosh^2 x} = \tanh^2 x$$

$$1-\operatorname{sech}^2 x = 1-\left(\frac{2}{e^x+e^{-x}}\right)^2 = \frac{e^{2x}+2+e^{-2x}-4}{\left(e^x+e^{-x}\right)^2} = \frac{e^{2x}+e^{-2x}-2}{e^{2x}+e^{-2x}+2} = \tanh^2 x$$

Both score B1M1A0 as we would need to see numerators and denominators factorised.

Note that we will allow an equivalent identity to be proved by exponentials and the given identity deduced e.g.

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$$

B1: Correct exponential form seen for cosh or sinh used

$$= \frac{e^{2x}}{4}+\frac{1}{2}+\frac{e^{-2x}}{4}-\frac{e^{2x}}{4}+\frac{1}{2}-\frac{e^{-2x}}{4} = 1$$

M1: Expands and collects terms

$$\Rightarrow \cosh^2 x - \sinh^2 x = 1 \Rightarrow 1-\operatorname{sech}^2 x = \tanh^2 x$$

A1*: Fully correct work leading to the correct identity

(b)	$\int \tanh^n 3x\,\mathrm{d}x = \int \tanh^{n-2} 3x \tanh^2 3x\,\mathrm{d}x$ $= \int \tanh^{n-2} 3x\left(1-\operatorname{sech}^2 3x\right)\mathrm{d}x$	Splits $\tanh^n 3x$ into $\tanh^{n-2} 3x \tanh^2 3x$ and applies $\tanh^2 3x = 1-\operatorname{sech}^2 3x$	M1
	Do **not** condone $\int \tanh^n 3x\,\mathrm{d}x = \int \tanh^{n-2} 3x \tanh^2 3x\,\mathrm{d}x$ $= \int \tanh^{n-2} 3x\left(1-\operatorname{sech}^2 x\right)\mathrm{d}x$ unless it is clear that $3x$ was intended and is therefore recovered in subsequent work.		
	$= \int \tanh^{n-2} 3x\,\mathrm{d}x - \int \tanh^{n-2} 3x \operatorname{sech}^2 3x\,\mathrm{d}x$ $\int \tanh^{n-2} 3x \operatorname{sech}^2 3x\,\mathrm{d}x = \frac{1}{3(n-1)} \tanh^{n-1} 3x$ Expands and integrates $\tanh^{n-2} 3x \operatorname{sech}^2 3x$ to obtain $\alpha \tanh^{n-1} 3x$ **Or it is possible to use parts for $\int \tanh^{n-2} 3x \operatorname{sech}^2 3x\,\mathrm{d}x$:** $\int \tanh^{n-2} 3x \operatorname{sech}^2 3x\,\mathrm{d}x = \frac{1}{3}\tanh 3x \tanh^{n-2} 3x - \frac{1}{3}\int 3(n-2)\tanh 3x \tanh^{n-3} 3x \operatorname{sech}^2 3x\,\mathrm{d}x$ $= \frac{1}{3}\tanh^{n-1} 3x - (n-2)\int \tanh^{n-2} 3x \operatorname{sech}^2 3x\,\mathrm{d}x$ $\Rightarrow \int \tanh^{n-2} 3x \operatorname{sech}^2 3x\,\mathrm{d}x = \frac{1}{3(n-1)} \tanh^{n-1} 3x$ To score it must be a complete method leading to $\alpha \tanh^{n-1} 3x$ as above		**d**M1
	$I_n = I_{n-2} - \frac{1}{3(n-1)}\left[\tanh^{n-1} 3x\right]_0^{\frac{1}{3}\ln 2} = I_{n-2} - \frac{1}{3(n-1)}\left(\frac{\mathrm{e}^{2\ln 2}-1}{\mathrm{e}^{2\ln 2}+1}\right)^{n-1}$ Introduces I_{n-2} and applies $x = \frac{1}{3}\ln 2$ using a correct exponential definition of tanh or accept use of a calculator if work is correct e.g. $\tanh(\ln 2) = \frac{3}{5}$		**dd**M1
	$I_n = I_{n-2} - \frac{\left(\frac{3}{5}\right)^{n-1}}{3(n-1)}$ but condone $I_n = I_{n-2} - \frac{\frac{3}{5}^{n-1}}{3(n-1)}$ Fully correct proof. Allow recovery from slips e.g. tanh →tan→tanh or e.g. $3x$ becoming x and then reverting to $3x$ again If there are clear errors that are not recovered score A0.		A1
			(4)

(c)	$I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)} = I_1 - \frac{\left(\frac{3}{5}\right)^{3-1}}{3(3-1)} - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$ Uses their reduction formula to obtain I_5 in terms of I_1 Note that there may have already been an attempt at I_1 Condone the use of the letter p for the $\frac{3}{5}$ and allow a "made up" p for this mark. This may be implied by e.g. $I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$, $I_3 = I_1 - \frac{\left(\frac{3}{5}\right)^{3-1}}{3(3-1)}$		M1
	$\int \tanh 3x \, dx = \frac{1}{3}\ln(\cosh 3x)$	Integrates to obtain $q\ln(\cosh rx)$ oe e.g. $q\ln(\text{sech } rx)$ Condone q and/or $r = 1$	M1
	$I_5 = \frac{1}{3}\ln\left(\frac{e^{\ln 2} + e^{-\ln 2}}{2}\right) - \frac{\left(\frac{9}{25}\right)}{6} - \frac{\left(\frac{81}{625}\right)}{12}$ Applies $x = \frac{1}{3}\ln 2$ using correct exponential definition of cosh or uses a calculator if work is correct e.g. $\cosh(\ln 2) = \frac{5}{4}$ to obtain a numerical expression for I_5 Must not be in terms of p now and must be using a value of p obtained in part (b)		**dd**M1
	$\frac{1}{3}\ln\frac{5}{4} - \frac{177}{2500}$	Correct answer in correct form (allow a =..., b =..., c =...) Allow –0.0708 for c	A1
			(4)
			Total 11

Note that part (c) is "Hence" so they need to be using the given reduction formula, however, it is possible to find I_3 directly e.g. :

$$I_5 = I_3 - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)}$$

$$\int \tanh^3 3x \, dx = \int \left(\tanh 3x - \tanh 3x \operatorname{sech}^2 3x\right) dx = \left[\frac{1}{3}\ln\left(\cosh 3x\right) + \frac{1}{6}\operatorname{sech}^2 3x\right]$$

Score **M1** for using the reduction formula to obtain I_5 in terms of I_3 (allow the letter p for the $\frac{3}{5}$ and allow a "made up" p for this mark) **and** then integrating $\tanh^3 3x$ to the correct form e.g. $\alpha\ln(\cosh 3x) + \beta\operatorname{sech}^2 3x$ (oe)

The second **M** mark would also score at this point as in the main scheme for integrating $\tanh 3x$ to obtain $q\ln(\cosh rx)$ oe e.g. $q\ln(\text{sech } rx)$

$$\left[\frac{1}{3}\ln\left(\cosh 3x\right) + \frac{1}{6}\operatorname{sech}^2 3x\right]_0^{\frac{1}{3}\ln 2} - \frac{\left(\frac{3}{5}\right)^{5-1}}{3(5-1)} = \frac{1}{3}\ln\frac{5}{4} + \frac{1}{6}\times\frac{16}{25} - \frac{1}{6} - \frac{27}{2500}$$

ddM1 for a complete method **using both limits** to obtain a numerical expression for I_5 using the correct exponential definitions or via a calculator.

A1: $\frac{1}{3}\ln\frac{5}{4} - \frac{177}{2500}$

Correct answer in correct form
(allow a =..., b =..., c =...) Allow –0.0708 for c